On the Transformation of Linear Differential Equations of the Second Order with Linear Coefficients.

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The transformation and reduction to a canonical form of the linear differential equation of the second order,

$$(A_0 x + B_0)\frac{d^2y}{dx^2} + (A_1 x + B_1)\frac{dy}{dx} + (A_2 x + B_2)\,y = 0 \tag{A}$$

has been studied by Weiler* and Schlömilch,† and again recently by Pochhammer,‡ in connection with his investigations on the integration of linear differential equations by means of definite integrals.

In the following pages I take up the problem anew, treating it, however, by methods of the *Theory of Invariants.*

The first thing necessary is, then, to find a *group of transformations* which transforms the given differential equation into one of the same type. For this purpose it is preferable to substitute for the differential equation (A) the two differential equations,§

$$\frac{d^2y}{dx^2} + 2\left(\frac{a}{x} + b\right)\frac{dy}{dx} + \left(\frac{f}{x^2} + \frac{2g}{x} + h\right)y = 0, \tag{B}$$

$$\frac{d^2y}{dx^2} + 2(a + bx)\frac{dy}{dx} + (f + 2gx + hx^2)\,y = 0, \tag{C}$$

which, combined, comprise the differential equation (A) as a special case.

* Crelle's Journal, Vol. 51 (1856), p. 123.

† Compendium der höheren Analysis: Braunschweig, 1879, Vol. II, p. 523.

‡ Mathematische Annalen, Vol. 38 (1891), pp. 225, 241.

§ This is the form in which Weiler considers the problem.

For if $A_0 \neq 0$, (A) can be reduced to the type (B) by the substitution $A_0x + B_0 = x'$; and if $A_0 = 0$, (A) is of the type (C).

The integrals of (A) are of a very different analytical character according as A_0 is $\neq 0$ or $= 0$; it seems, therefore, indeed more natural to separate the essentially different cases rather than to artificially unite them in one canonical form.

§1.

The Differential Equation (B).

a). *Group of Transformations.*

The differential equation (B) is transformed by the substitution

$$x = \varkappa x', \quad y = \nu x'^{\lambda} e^{\mu x'} y', \quad (\varkappa \neq 0) \tag{1}$$

into a differential equation of the same type, viz.

$$\frac{d^2y'}{dx'^2} + 2\left(\frac{a'}{x'} + b'\right)\frac{dy'}{dx'} + \left(\frac{f'}{x'^2} + \frac{2g'}{x'} + h'\right)y' = 0, \tag{B$'$}$$

where

$$\left.\begin{aligned} a' &= a + \lambda, \\ b' &= \varkappa b + \mu, \\ f' &= \lambda(\lambda - 1) + 2a\lambda + f, \\ g' &= \varkappa g + \varkappa\lambda b + \mu a + \lambda\mu, \\ h' &= \varkappa^2 h + 2\varkappa\mu b + \mu^2. \end{aligned}\right\} \tag{2}$$

The entire system of all transformations of the form (1) constitute a *group* $\mathfrak{G}$; to every transformation of the group there exists an "inverse" transformation in the group. Two differential equations (B) and (B′) will be said to be *equivalent* with respect to the group $\mathfrak{G}$, if they can be transformed into one another by transformations of the group. In order that (B) and (B′) shall be equivalent, it is necessary and sufficient that there exist values of the parameters $\varkappa$, λ, μ which satisfy the system (2), and that moreover $\varkappa \neq 0$.

b). *Invariants.*

The elimination of λ, μ between the equations (2) leads to the following result:

The Differential Equation (B) *has, with respect to the group* $\mathfrak{G}$, *an* ABSOLUTE INVARIANT,

$$A = f + a - a^2 \tag{3}$$

and TWO INVARIANTS,

$$\left.\begin{aligned} B &= g - ab, \\ C &= h - b^2. \end{aligned}\right\} \tag{4}$$

Their invariant character is expressed by the equations

$$\left.\begin{aligned} A' &= A, \\ B' &= \varkappa B, \\ C' &= \varkappa^2 C. \end{aligned}\right\} \tag{5}$$

if A', B', C' denote the same expressions formed with the coefficients of the transformed differential equation (B′).

c). *Canonical forms and conditions of equivalence.*

The transformation

$$x = \varkappa z, \quad y = z^{-a} e^{-\kappa b z} v \tag{6}$$

of our group reduces (B) to

$$\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2\varkappa B}{z} + \varkappa^2 C\right) v = 0. \tag{7}$$

Hence we obtain, by a proper choice of $\varkappa$, the following canonical forms:

I. *Case:* $B \neq 0$, $C \neq 0$.

$$\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} + I\right) v = 0, \tag{8}$$

where I denotes the absolute invariant

$$I = \frac{C}{B^2}. \tag{9}$$

II. *Case:* $B \neq 0$, $C = 0$.

$$\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z}\right) v = 0. \tag{10}$$

III. *Case:* $B = 0$, $C \neq 0$.

$$\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + 1\right) v = 0. \tag{11}$$

IV. *Case:* $B=0,\quad C=0$.

$$\frac{d^2v}{dz^2}+\frac{A}{z^2}=0. \tag{12}$$

Hence follows the theorem:

In order that two differential equations (B) *and* (B′) *shall be equivalent, it is necessary and sufficient that*

$$a)\quad A'=A$$

and that $b)$ *a quantity* $\varkappa \neq 0$
can be so determined that

$$B'=\varkappa B,$$
$$C'=\varkappa^2 C.$$

d). Other canonical forms, preferable for integration by series.

From the general theory of linear differential equations it follows that (B) can be integrated by a permanently convergent *power-series* of the form

$$y=\sum_{\nu=0}^{\infty} c_\nu x^{r+\nu},\qquad (c_0 \neq 0),$$

where r is a root of the *indicial equation,**

$$f_0(r)=r(r-1)+2ra+f=0. \tag{13}$$

In passing, we notice that its discriminant is an absolute invariant, viz.

$$\Delta=\frac{1}{4}-A. \tag{14}$$

The coefficients c_ν are determined by the recurrent formula†

$$c_\nu f_0(r+\nu)+c_{\nu-1}f_1(r+\nu-1)+c_{\nu-2}f_2(r+\nu-2)=0, \tag{15}$$

where

$$f_1(r)=2(g+br),\quad f_2(r)=h.$$

We propose to use the transformation (1) to reduce the relation (15) to only *two terms*. This can always be obtained by making $h'=0$ by a proper choice

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* "Determinierende Fundamentalgleichung (Fuchs); see Craig, A Treatise on Linear Differential Equations, p. 118.

† Fuchs, Crelle's Journal, Vol. 66; Frobenius, Crelle's Journal, Vol. 76.

of μ; at the same time we can choose λ so that $f' = 0$. This leads to the following canonical forms:

a). *Case* I *and* III: $C \neq 0$.

$$\xi \frac{d^2\eta}{d\xi^2} = (\xi - \rho) \frac{d\eta}{d\xi} + \alpha\eta, * \tag{16}$$

where

$$\rho = 1 + \sqrt{1 - 4A}, \quad \alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4A} - \frac{B}{\sqrt{-C}}.$$

b). *Case* II: $B \neq 0$, $C = 0$.

$$\xi \frac{d^2\eta}{d\xi^2} + \rho \frac{d\eta}{d\xi} - \eta = 0. \dagger \tag{17}$$

c). *Case* IV: $B = 0$, $C = 0$.

$$\xi \frac{d^2\eta}{d\xi^2} + \rho \frac{d\eta}{d\xi} = 0. \tag{18}$$

In (17) and (18) ρ has the same value as in (16).

The reduction of (15) to two terms may, however, be performed still in other ways which offer certain advantages.

If $C \neq 0$, we can determine λ and μ so that

$$g' = 0 \text{ and } h' = 0,$$

since the resultant obtained by eliminating μ is

$$C(\lambda + a)^2 + B^2 = 0.$$

Hence the canonical form

$$\frac{d^2u}{dt^2} + \left(\frac{1 - \lambda' - \lambda''}{t} - 1\right)\frac{du}{dt} + \frac{\lambda'\lambda''}{t^2} u = 0, \tag{19}$$

$$\left.\begin{matrix}\lambda' \\ \lambda''\end{matrix}\right\} = \frac{1}{2} - \frac{B}{\sqrt{-C}} \pm \frac{1}{2}\sqrt{1 - 4A}.$$

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* This is, apart from the notation, Weiler's canonical form; see l. c., p. 127; also Schlömilch, l. c., p. 531; and Pochhammer, Math. Ann., Vol. 36, p. 84.

† Pochhammer, Math. Ann., Vol. 38, p. 226.

This canonical form is symmetric with respect to the two fundamental integrals; they are, if $\lambda' - \lambda''$ is not an integer,

$$\left.\begin{aligned} u_1 &= t^{\lambda'} F(\lambda';\ 1 + \lambda' - \lambda'';\ t), \\ u_2 &= t^{\lambda''} F(\lambda'';\ 1 + \lambda'' - \lambda';\ t), \end{aligned}\right\} \tag{20}$$

where we denote, with Pochhammer,* by $F(a; r; t)$ the permanently convergent series

$$F(a;\ r;\ t) = 1 + \frac{a}{1.r} t + \frac{a(a+1)}{1.2r(r+1)} t^2 + \dots . \tag{21}$$

The two canonical forms (16) and (19) are transformable into one another by the substitution

$$t = \xi, \quad u = t^{\lambda'} \eta .$$

If, in particular, $B = 0$, while as before $C \neq 0$, the relation (13) may be reduced to two terms by making

$$f_1(r) \equiv 0 .$$

For, after having determined μ so that $b' = 0$, it follows from $B = 0$ that also $g' = 0$. Moreover, the parameter λ may be used to make $a' = \dfrac{1}{2}$ or to make $f' = 0$. Accordingly we obtain the two equivalent canonical forms

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} + \left(1 - \frac{n^2}{t^2}\right) u = 0, \tag{22}$$

(*Bessel's Equation.*)

or

$$\frac{d^2u}{dt^2} + \frac{2n+1}{t}\frac{du}{dt} + u = 0, \tag{23}$$

$$n^2 = \frac{1}{4} - A .$$

In concluding we give the necessary and sufficient condition that our differential equation (B) shall be reducible, by a transformation of our group, to the form

$$\frac{d^2y'}{dx'^2} + 2\left(\frac{a'}{x'} + b'\right)\frac{dy'}{dx'} = 0, \tag{24}$$

in which case it is integrable by quadratures; the condition is

$$(AC - B^2)^2 + B^2 C = 0 . \tag{25}$$

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* Math. Annalen, Vol. 36, p. 84.

§2.

The Differential Equation (C).

a). Group of transformations.

The differential equation

$$\frac{d^2y}{dx^2} + 2(a + bx)\frac{dy}{dx} + (f + 2gx + hx^2)\,y = 0 \tag{C}$$

is transformed by the transformation

$$x = \varkappa x' + \lambda,\quad y = e^{\mu x' + \frac{\nu}{2}x'^2}\,y', \qquad (\varkappa \neq 0) \tag{1}$$

into

$$\frac{d^2y'}{dx'^2} + 2(a' + b'x')\frac{dy'}{dx'} + (f' + 2g'x' + h'x'^2)\,y' = 0, \tag{C'}$$

where

$$\left.\begin{aligned} a' &= \varkappa\,(a + b\lambda) + \mu, \\ b' &= \varkappa^2 b + \nu, \\ f' &= \varkappa^2(f + 2g\lambda + h\lambda^2) + 2\mu\varkappa\,(a + b\lambda) + \mu^2 + \nu, \\ g' &= \varkappa^3(g + \lambda h) + \mu\varkappa^2 b + \nu\varkappa\,(a + b\lambda) + \mu\nu, \\ h' &= \varkappa^4 h + 2\nu\varkappa^2 b + \nu^2. \end{aligned}\right\} \tag{2}$$

The transformations (1) form again a group, $\mathfrak{H}$, and the same conclusions can be applied as in §1.

b). Invariants and covariants.

The differential equation (C) has, with respect to the group $\mathfrak{H}$, *two invariants,*

$$\left.\begin{aligned} H &= h - b^2, \\ D &= (h - b^2)(f - a^2 - b) - (g - ab)^2, \end{aligned}\right\} \tag{3}$$

characterized by

$$H' = \varkappa^4 H,\quad D' = \varkappa^6 D. \tag{4}$$

But these covariants are not sufficient to distinguish between the non-equivalent cases. It seems necessary to resort to a *covariant-criterion.* If we denote

$$\left.\begin{aligned} F &= f - a^2 - b, & F' &= f' - a'^2 - b', \\ G &= g - ab, & G' &= g' - a'b', \\ H &= h - b^2, & H' &= h' - b'^2, \end{aligned}\right\} \tag{5}$$

the function

$$\Phi(x) = F + 2Gx + Hx^2 \tag{6}$$

has the characteristic property of a covariant, viz.

$$F' + 2G'x' + H'x'^2 = \varkappa^2 (F + 2Gx + Hx^2),$$

or shorter,

$$\Phi'(x') = \varkappa^2 \Phi(x), \tag{7}$$

as is seen from the following relations:

$$\left.\begin{aligned} F' &= \varkappa^2 F + 2\varkappa^2 \lambda G + \varkappa^2 \lambda^2 H, \\ G' &= \qquad\qquad \varkappa^3 G + \varkappa^3 \lambda H, \\ H' &= \qquad\qquad\qquad\qquad \varkappa^4 H. \end{aligned}\right\} \tag{8}$$

c). *Reduction to canonical forms and conditions of equivalence.*

By the transformation

$$y = e^{-\left(ax + b\frac{x^2}{2}\right)} v \tag{9}$$

(which is one of our group $\mathfrak{H}$), the differential equation (C) is reduced to

$$\frac{d^2v}{dx^2} + (F + 2Gx + Hx^2) v = 0. \tag{10}$$

This differential equation can be further reduced by a transformation $x = \varkappa z + \lambda$. The following canonical forms are obtained:

I. *Case:* $H \neq 0$.

$$\frac{d^2v}{dz^2} + (\rho + z^2) v = 0, \tag{11}$$

$$\rho = \sqrt{I},$$

if we denote by I the absolute invariant

$$I = \frac{D^2}{H^3}. \tag{12}$$

II. *Case:* $H=0$, $D \neq 0$.

It follows: $G \neq 0$, hence we can make $F'=0$, $G'=-\frac{1}{3}$,

$$\frac{d^2v}{dz^2}-\frac{1}{3}zv=0, \tag{13}$$

(*Scherk*–Lobatto's† equation.*)

III. *Case:* $H=0$, $D=0$.

It follows: $G=0$.

1. *Subcase:* $\Phi(x) \not\equiv 0$.

It follows: $F \neq 0$, hence we may make $F'=1$,

$$\frac{d^2v}{dz^2}+v=0. \tag{14}$$

2. *Subcase:* $\Phi(x) \equiv 0$; that is, $F=0$, $G=0$, $H=0$, hence

$$\frac{d^2v}{dz^2}=0. \tag{15}$$

Hence the result:

In order that two differential equations (C) *and* (C′) *shall be equivalent, it is necessary and sufficient that*

a). A quantity $\varkappa \neq 0$ can so be determined that $H'=\varkappa^4 H$, $D'=\varkappa^6 D$.

b). That the two covariants $\Phi(x)$ and $\Phi'(x')$ are either both identically zero, or both not identically zero.

In case I, a different canonical form is preferable for purposes of integration by series or by definite integrals. If $H \neq 0$, $\varkappa$, λ, μ, ν can be so determined that

$$a'=0,\ g'=0,\ h'=0,\ b'=-\frac{1}{2}$$

(use (8)). This leads to the canonical form

$$\frac{d^2\eta}{d\xi^2}-\xi\frac{d\eta}{d\xi}-\alpha\eta=0,\ddagger \tag{16}$$

$$\alpha=\frac{1}{2}+\frac{1}{2}\sqrt{-I}.$$

* Crelle's Journal, Vol. 10, p. 92.

† Crelle's Journal, Vol. 17, p. 363; see also Pochhammer, Math. Ann., Vol. 38, pp. 242, 247.

‡ Weiler, l. c., p. 128, and Pochhammer, Math. Ann., Vol. 38, p. 241.

Finally we mention the following special cases:

If $H \neq 0$, $D = 0$, (11) becomes

$$\frac{d^2u}{dz^2} + t^2v = 0. \tag{17}$$

(Special case of *Riccati's equation.*)

$I = -1$ is the necessary and sufficient condition that (C) shall be reducible to the form

$$\frac{d^2y'}{dx'^2} + 2(a' + b'x')\frac{dy'}{dx'} = 0. \tag{18}$$

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